Exact two-holon wave functions in the Kuramoto–Yokoyama model

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Abstract

We construct the explicit two-holon eigenstates of the SU(2) Kuramoto–Yokoyama model at the level of explicit wave functions. We derive the exact energies and obtain the individual holon momenta, which are quantized according to the half-Fermi statistics of the holons.

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I. INTRODUCTION

One milestone towards the understanding of fractional quantization in one dimension is the $1/r^2$ model independently introduced by Haldane [1] and Shastry [2] in 1988. The model describes a spin 1/2 chain with a Heisenberg interaction which falls off as one over the square of the distance between the sites. The exact ground state is provided by a trial wave function proposed by Gutzwiller [3] as early as in 1963. The Haldane–Shastry Model (HSM) offers the opportunity of studying spinons, *i.e.*, the elementary excitations of one-dimensional spin chains, on the level of explicit and analytical expressions for one and two-spinon wave functions [4], which are at least at present not available for any other model. Kuramoto and Yokoyama [5] generalized the model to allow for mobile holes (*i.e.*, empty lattice sites) with a hopping parameter that also falls off with $1/r^2$ as a function of the distance. The Kuramoto–Yokoyama Model (KYM) hence contains spin and charge degrees of freedom, and accordingly supports spinon and holon excitations, which carry spin $\frac{1}{2}$ but no charge and charge $+1$ but no spin, respectively. In principle, the KYM allows for a similarly explicit construction of holon wave functions, which so far have only been obtained for states involving a single holon. The reason for this deficit has been of technical nature, related to the commutation relations of the operators used to build the Hilbert space of these fractionally quantized excitations. Whereas for the spinons one can use bosonic spin-flip operators, one needs fermionic creation and annihilation operators for the holons.

In this article we address and overcome this technical problem as we construct the explicit wave functions for two-holon excitations of the KYM. The article is organized as follows: In Section II we review the KYM and its properties. In Section III and IV, we briefly discuss the ground state at half filling and the spinon excitations. We further review the analytic results so far known for the one-holon excitations in Section V as a preliminary for the construction of the explicit two-holon wave functions to be done in Section VI. Therein we derive the exact energies and individual holon momenta, which turn out to be quantized according to half-Fermi statistics of the holons.

II. KURAMOTO–YOKOYAMA MODEL

The Kuramoto–Yokoyama model [5] is most conveniently formulated by embedding the one-dimensional chain with periodic boundary conditions into the complex plane by mapping it onto the unit circle with the sites located at complex positions $\eta_\alpha = \exp\big(i\frac{2\pi}{N}\alpha\big)$, where N denotes the number of sites and $\alpha = 1, \ldots, N$. The sites can be either singly occupied by an up or down-spin electron or empty. The Hamiltonian is given by

$$H_{\mathrm{KY}} = -\frac{\pi^2}{N^2} \sum_{\substack{\alpha,\beta=1 \\ \alpha\neq\beta}}^{N} \frac{P_{\alpha\beta}}{|\eta_\alpha - \eta_\beta|^2}, \tag{1}$$

where $P_{\alpha\beta}$ exchanges the configurations on the sites η_α and η_β including a minus sign if both are fermionic. Rewriting (1) in terms of spin and electron creation and annihilation operators yields

$$H_{\mathrm{KY}} = \frac{2\pi^2}{N^2} \sum_{\alpha\neq\beta}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} P_{\mathrm{G}} \left[-\frac{1}{2} \sum_{\sigma=\uparrow\downarrow} \left(c^\dagger_{\alpha\sigma} c_{\beta\sigma} + c^\dagger_{\beta\sigma} c_{\alpha\sigma} \right) + \vec{S}_\alpha \cdot \vec{S}_\beta - \frac{n_\alpha n_\beta}{4} + n_\alpha - \frac{1}{2} \right] P_{\mathrm{G}}, \tag{2}$$

where the Gutzwiller projector $P_{\mathrm{G}} = \prod_\alpha (1 - c^\dagger_{\alpha\uparrow} c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow} c_{\alpha\uparrow})$ enforces at most single occupancy on all sites. The charge occupation and spin operators are given by $n_\alpha = c^\dagger_{\alpha\uparrow} c_{\alpha\uparrow} + c^\dagger_{\alpha\downarrow} c_{\alpha\downarrow}$ and $S^a_\alpha = \frac{1}{2} \sum_{\sigma,\sigma'} c^\dagger_{\alpha\sigma} \tau^a_{\sigma\sigma'} c_{\alpha\sigma'}$, where τ^a, $a = x, y, z$, denote the Pauli matrices.

The interaction strength in (1) is an analytic function of the lattice sites by use of

$$\frac{1}{|\eta_\alpha - \eta_\beta|^2} = -\frac{\eta_\alpha \eta_\beta}{(\eta_\alpha - \eta_\beta)^2}. \tag{3}$$

The KYM is supersymmetric, *i.e.*, the Hamiltonian (1) commutes with the operators $J^{ab} = \sum_\alpha a^\dagger_{\alpha a} a_{\alpha b}$, where $a_{\alpha a}$ denotes the annihilation operator of a particle of species a (a runs over up- and down-spin as well as empty site) at site η_α. The traceless parts of the operators J^{ab} generate the Lie superalgebra su(1|2), which includes in particular the total spin $\vec{S} = \sum_{\alpha=1}^{N} \vec{S}_\alpha$. In addition, the KYM possesses a super-Yangian symmetry [6], which causes its amenability to rather explicit solution.

III. VACUUM STATE

We first review the ground state at half filling, which is the state containing no excitations (neither spinons nor holons). For N even, this vacuum state is constructed by the

Gutzwiller projection of a filled band (or Slater determinant (SD) state) containing a total of N electrons:

$$|\Psi_0\rangle = P_{\rm G} \prod_{|q|<q_{\rm F}} c^\dagger_{q\uparrow} c^\dagger_{q\downarrow} |0\rangle \equiv P_{\rm G} \left|\Psi_{\rm SD}^N\right\rangle . \tag{4}$$

Taking the fully polarized state $|0_\downarrow\rangle = \prod_\alpha c^\dagger_{\alpha\downarrow} |0\rangle$ as reference state, we can rewrite the vacuum state as

$$|\Psi_0\rangle = \sum_{\{z_i\}} \Psi_0(z_1, \ldots, z_M)\, S^+_{z_1} \ldots S^+_{z_M} |0_\downarrow\rangle , \tag{5}$$

where $M = N/2$ and the z_i's denote the up-spin coordinates. The sum in (5) extends over all possible ways to distribute the coordinates z_i's over the lattice sites η_α. The wave function is given by [1, 2]

$$\Psi_0(z_1, \ldots, z_M) = \prod_{i<j}^M (z_i - z_j)^2 \prod_{i=1}^M z_i, \tag{6}$$

its energy is

$$E_0 = -\frac{\pi^2}{4N}. \tag{7}$$

The total momentum of a state is evaluated by considering the operator $\mathbf{T}$, which translates all coordinates counterclockwise by one site. $\mathbf{T}$ is related to the momentum operator $\mathbf{P}$ via

$$\mathbf{T} = \exp(-i\mathbf{P}). \tag{8}$$

This yields the momentum of $|\Psi_0\rangle$ to equal zero if N is divisible by four and π otherwise.

Note that (6) represents the ground state of (1) only at half filling, *i.e.*, when all sites are occupied. As was shown by Kuramoto and Yokoyama [5], the ground state away from half-filling can be constructed by Gutzwiller projection similar to (4).

IV. SPINON EXCITATIONS

Let N be odd and $M = (N-1)/2$. A localized spinon at site "η_γ" is constructed by the Gutzwiller projection of an electron inserted in a Slater determinant state of $N+1$ electrons:

$$\left|\Psi_\gamma^{\rm sp}\right\rangle = P_{\rm G}\, c_{\gamma\downarrow} \left|\Psi_{\rm SD}^{N+1}\right\rangle . \tag{9}$$

The annihilation of the electron causes an inhomogeneity in the spin and charge degree of freedom. After the projection, however, only the inhomogeneity in the spin survives. The

spinon hence possesses spin one-half but no charge. The wave function of a localized spinon is given by [7]

$$\Psi_{\gamma}^{\text{sp}}(z_1,\ldots,z_M)=\prod_{i=1}^{M}(\eta_\gamma-z_i)\;\Psi_0(z_1,\ldots,z_M), \tag{10}$$

where Ψ_0 is defined in (5). Fourier transformation yields the momentum eigenstates

$$\left|\Psi_m^{\text{sp}}\right\rangle=\sum_{\alpha=1}^{N}(\bar{\eta}_\gamma)^m\left|\Psi_\gamma^{\text{sp}}\right\rangle, \tag{11}$$

which vanish identically unless $0\leq m\leq M$. In particular, this implies that the localized one-spinon states (9) form an overcomplete set. It is hence not possible to interpret the "coordinate" η_γ literally as the position of the spinon. The momentum eigenstates (11) are found to be exact energy eigenstates of the KYM, with its energies given by [7]

$$E_m^{\text{sp}}=\frac{2\pi^2}{N^2}\left(\frac{N-1}{2}-m\right)m. \tag{12}$$

The spinons obey half-Fermi statistics, which was first found by the investigation of their state counting rules [8]. Later it became apparent that the fractional statistics of the spinons manifests itself in the quantization rules for the individual spinon momenta as well [9, 10].

V. ONE-HOLON EXCITATIONS

The charged elementary excitations of the model are holons, the concept of which must be invoked whenever holes and thereby charge carries are doped into the chain. A localized holon at lattice site η_ξ is constructed as

$$\left|\Psi_\xi^{\text{ho}}\right\rangle=c_{\xi\downarrow}P_{\text{G}}\,c_{\xi\downarrow}^{\dagger}\left|\Psi_{\text{SD}}^{N-1}\right\rangle. \tag{13}$$

(Alternatively we could use the operators $c_{\xi\uparrow}$ and $c_{\xi\uparrow}^{\dagger}$.) Compared to the spinon we eliminate the inhomogeneity in spin while creating an inhomogeneity in the charge distribution after Gutzwiller projection. Thus the holon has no spin but charge $e>0$ (as the electron charge at site η_ξ is removed). Note that the holon is strictly localized at the holon coordinate ξ, as holon states on neighboring coordinates are orthogonal. In total, there are N independent one-holon states (13).

Momentum eigenstates are constructed from (13) by Fourier transformation. It turns out that only $(N+3)/2$ of them are energy eigenstates [11]. We will restrict ourselves to this

subset in the following. These states are readily described in terms of their wave functions. We take $|0_\downarrow\rangle$ as reference state, and write the one-holon energy eigenstates as [11]

$$\left|\Psi_m^{\mathrm{ho}}\right\rangle = \sum_{\{z_i;h\}} \Psi_m^{\mathrm{ho}}(z_1,\ldots,z_M;h)\, c_{h\downarrow}\, S_{z_1}^+ \ldots S_{z_M}^+ \left|0_\downarrow\right\rangle, \tag{14}$$

where the sum extends over all possible ways to distribute the up-spin coordinates z_i and the holon coordinate h over the lattice sites η_α subject to the restriction $z_i \neq h$. The one-holon wave function is given by

$$\Psi_m^{\mathrm{ho}}(z_1,\ldots,z_M;h) = h^m \prod_{i=1}^{M} (h - z_i)\, \Psi_0(z_1,\ldots,z_M), \tag{15}$$

where Ψ_0 is given by (6). Note that as a sum over the coordinates h is included in (14), no such sum is required in (15). It can be shown that $0 \leq m \leq M+1$, where $M = (N-1)/2$ is the number of up-spin coordinates, the wave function (15) represents an exact energy eigenstate with energy [11]

$$E_m = \frac{2\pi^2}{N^2} \left(m - \frac{N+1}{2} \right) m. \tag{16}$$

For other values of m, the states $\left|\Psi_m^{\mathrm{ho}}\right\rangle$ do not vanish identically (as $|\Psi_m^{\mathrm{sp}}\rangle$ for spinons do), but are not eigenstates of the Kuramoto–Yokoyama Hamiltonian (1) either. Consequently, we are allowed to refer to the states (14) with (15) as "holons" only if $0 \leq m \leq M+1$.

Note that this implies that the states (13) do not really constitute "holons" localized in position space, but only basis states which can be used to construct holons if the momentum is chosen adequately. The total number of single-holon states is given by $M+2$, according to the number of distinct values m is allowed to assume. Since the states (13) are orthogonal for different lattice positions ξ, there are $N = 2M+1$ orthogonal position basis states $\left|\Psi_\xi^{\mathrm{ho}}\right\rangle$. Hence the states $\left|\Psi_\xi^{\mathrm{ho}}\right\rangle$ cannot strictly be holons, but rather constitute incoherent superpositions of holons and other states. It is clear from these considerations that it is not possible to localize a holon onto a single lattice site. The best we can do is to take a Fourier transform of the exact eigenstates $\left|\Psi_m^{\mathrm{ho}}\right\rangle$ for $0 \leq m \leq M+1$ back into position space. The resulting "localized" holon states will be true holons but will not be localized strictly onto lattice sites. Such a true holon state "localized" at a given lattice site will not be orthogonal to such a state "localized" at the neighboring lattice site, as there are only $M+2$ holon states while there are N lattice sites. The situation is hence very similar to the case of

the spinons, which form an overcomplete set and are well known to be non-orthogonal if "localized" on neighboring lattice sites.

The one-holon momenta of the states (14) with (15) are derived in analogy to the vacuum state to be

$$p_m^{\text{ho}} = \frac{\pi}{2}N + \frac{2\pi}{N}\left(m - \frac{1}{4}\right) \mod 2\pi. \tag{17}$$

If we introduce the one-holon dispersion

$$\epsilon^{\text{ho}}(p) = -\frac{1}{2}\left(\frac{\pi^2}{4} - p^2\right) - \frac{\pi^2}{8N^2}, \quad -\frac{\pi}{2} \leq p \leq \frac{\pi}{2}, \tag{18}$$

we can rewrite (16) with the vacuum energy (7) as

$$E_m = E_0 + \epsilon^{\text{ho}}(p_m^{\text{ho}}). \tag{19}$$

VI. TWO-HOLON EXCITATIONS

A. Momentum eigenstates

Let N be even and $M = (N-2)/2$. The two-holon state with holons localized at η_{ξ_1} and η_{ξ_2} is constructed in analogy to (13) as

$$\left|\Psi_{\xi_1\xi_2}^{\text{ho}}\right\rangle = c_{\xi_1\downarrow}c_{\xi_2\downarrow}P_{\text{G}}\, c_{\xi_1\downarrow}^\dagger c_{\xi_2\downarrow}^\dagger \left|\Psi_{\text{SD}}^{N-2}\right\rangle. \tag{20}$$

In analogy to (15), a momentum basis for the two-holon eigenstates is provided by the wave functions

$$\Psi_{mn}^{\text{ho}}(z_1,\ldots,z_M;h_1,h_2) = (h_1-h_2)(h_1^m h_2^n + h_1^n h_2^m)\prod_{i=1}^{M}(h_1-z_i)(h_2-z_i)\Psi_0(z_1,\ldots,z_M), \tag{21}$$

where Ψ_0 is again given by (6), $h_{1,2}$ denote the holon coordinates, and the integers m and n satisfy

$$0 \leq n \leq m \leq M+1. \tag{22}$$

The corresponding state is then given by

$$\left|\Psi_{mn}^{\text{ho}}\right\rangle = \sum_{\{z_i;h_1,h_2\}} \Psi_{mn}^{\text{ho}}(z_1,\ldots,z_M;h_1,h_2)\, c_{h_1\downarrow}\, c_{h_2\downarrow}\, S_{z_1}^+ \ldots S_{z_M}^+ \left|0_\downarrow\right\rangle, \tag{23}$$

where the sum extends over all possible ways to distribute the up-spin coordinates z_i and the holon coordinates $h_{1,2}$ over the lattice sites η_α subject to the restriction $z_i \neq h_1 \neq h_2$. The momentum of the states (23) is easily found to be

$$p_{mn}^{\text{ho}} = \frac{\pi}{2}N + \frac{2\pi}{N}\big(m+n\big) \mod 2\pi. \tag{24}$$

It can further be shown that the states (23) are spin singlets, *i.e.*, they are annihilated by $S^{\pm}$ as well as S^z.

B. Action of $H_{\mathbf{KY}}$ on the momentum eigenstates

In the following we will construct the two-holon energy eigenstates starting from (21). First, we introduce the auxiliary wave functions

$$\varphi_{mn}(z_1,\ldots,z_M;h_1,h_2) = h_1^m h_2^n \prod_{i=1}^{M}(h_1-z_i)(h_2-z_i)\,\Psi_0(z_1,\ldots,z_M). \tag{25}$$

The action of the Hamiltonian on the states (21) will be obtained later via

$$\Psi_{mn}^{\text{ho}} = \varphi_{m+1,n} + \varphi_{n+1,m} - \varphi_{m,n+1} - \varphi_{n,m+1}. \tag{26}$$

Second, we rewrite the Hamiltonian (2) in analogy to [11] as

$$H_{\text{KY}} = \frac{2\pi^2}{N^2}\left(H_{\text{S}}^{\text{ex}} + H_{\text{S}}^{\text{Is}} + H_{\text{V}} + H_{\text{C}}^{\uparrow} + H_{\text{C}}^{\downarrow}\right), \tag{27}$$

where we separate the spin-exchange, spin-Ising, potential, $\uparrow$-charge kinetic term, and $\downarrow$-charge kinetic terms. In the following we treat each term separately.

For the spin-exchange term we begin by observing that $[S_\alpha^+ S_\beta^- \varphi_{nm}](z_1,\ldots,z_M;h_1,h_2)$ is identically zero unless one of the arguments $z_1,\ldots,z_M$ equals η_α. We have

$$\begin{aligned}
\Big[H_{\text{S}}^{\text{ex}}\varphi_{mn}\Big](z_1,\ldots,z_M;h_1,h_2) &\equiv \left[\sum_{\alpha\neq\beta}^{N}\frac{P_{\text{G}}S_\alpha^+S_\beta^-P_{\text{G}}}{|\eta_\alpha-\eta_\beta|^2}\varphi_{mn}\right](z_1,\ldots,z_M;h_1,h_2)\\
&= \sum_{j=1}^{M}\sum_{\beta\neq j}^{N}\frac{\eta_\beta}{|z_j-\eta_\beta|^2}\frac{\varphi_{nm}(z_1,\ldots,z_{j-1},\eta_\beta,z_{j+1},\ldots,z_M;h_1,h_2)}{\eta_\beta}\\
&= \sum_{j=1}^{M}\sum_{l=0}^{N-1}\sum_{\beta\neq j}^{N}\frac{\eta_\beta(\eta_\beta-z_j)^l}{l!|z_j-\eta_\beta|^2}\frac{\partial^l}{\partial z_j^l}\left(\frac{\varphi_{nm}(z_1,\ldots,z_M;h_1,h_2)}{z_j}\right)\\
&= \sum_{j=1}^{M}\sum_{l=0}^{N-1}A_l\frac{z_j^{l+1}}{l!}\frac{\partial^l}{\partial z_j^l}\frac{\varphi_{mn}}{z_j}
\end{aligned}$$

$$= \sum_{j=1}^{M} \left(\frac{(N-1)(N-5)}{12} z_j - \frac{N-3}{2} z_j^2 \frac{\partial}{\partial z_j} + \frac{1}{2} z_j^3 \frac{\partial^2}{\partial z_j^2} \right) \frac{\varphi_{mn}}{z_j}$$

$$= \left\{ \frac{M}{12}(5-2N) h_1^m h_2^n - h_1^m h_2^n \sum_{i \neq j}^{M} \frac{1}{|z_i - z_j|^2} - h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2} - h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} \right.$$
$$\left. + \frac{N-3}{2} \left(h_1^m h_2^{n+1} \frac{\partial}{\partial h_2} + h_1^{m+1} h_2^n \frac{\partial}{\partial h_1} \right) + \frac{h_1^m h_2^{n+2}}{h_1 - h_2} \frac{\partial}{\partial h_2} - \frac{h_1^{m+2} h_2^n}{h_1 - h_2} \frac{\partial}{\partial h_1} \right\} \frac{\varphi_{mn}}{h_1^m h_2^n}, \quad (28)$$

where we have introduced the coefficients $A_l = -\sum_{\alpha=1}^{N-1} \eta_\alpha^2 (\eta_\alpha - 1)^{l-2}$. Evaluation of the latter yields $A_0 = (N-1)(N-5)/12$, $A_1 = -(N-3)/2$, $A_2 = 1$, and $A_l = 0$ for $2 < l \leq N-1$ [4].

For the spin-Ising term we obtain

$$\left[H_{\mathrm{S}}^{\mathrm{Is}} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2) \equiv \left[\sum_{\alpha \neq \beta}^{N} \frac{P_{\mathrm{G}} S_\alpha^z S_\beta^z P_{\mathrm{G}}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2)$$
$$= \left\{ \sum_{i \neq j}^{M} \frac{1}{|z_i - z_j|^2} + \sum_{i=1}^{M} \frac{1}{|z_i - h_1|^2} + \sum_{i=1}^{M} \frac{1}{|z_i - h_2|^2} + \frac{1/2}{|h_1 - h_2|^2} \right.$$
$$\left. - N \frac{N^2 - 1}{48} \right\} \varphi_{mn}. \quad (29)$$

The potential term yields

$$\left[H_{\mathrm{V}} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2) \equiv \left[\sum_{\alpha \neq \beta}^{N} \frac{P_{\mathrm{G}} \left(-\frac{1}{4} n_\alpha n_\beta + n_\alpha - \frac{1}{2} \right) P_{\mathrm{G}}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2)$$
$$= \left\{ -\frac{1}{2} \frac{1}{|h_1 - h_2|^2} - \frac{N^2 - 1}{12} + \frac{N}{4} \frac{N^2 - 1}{12} \right\} \varphi_{mn}. \quad (30)$$

The charge kinetic terms deserve particular care as new techniques are required.

For the $\uparrow$-charge kinetic term, we first observe that $[c_{\beta\uparrow} c_{\alpha\uparrow}^\dagger \varphi_{mn}](z_1, \ldots, z_M; h_1, h_2)$ is identically zero unless one of the arguments $z_1, \ldots, z_M$ equals η_α. We thus find

$$\left[H_{\mathrm{C}}^{\uparrow} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2) \equiv \left[\sum_{\alpha \neq \beta}^{N} \frac{P_{\mathrm{G}} c_{\beta\uparrow} c_{\alpha\uparrow}^\dagger P_{\mathrm{G}}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right](z_1, \ldots, z_M; h_1, h_2)$$
$$= \sum_{\alpha = h_1, h_2} \sum_{\beta \neq \alpha}^{N} \frac{1}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn}$$
$$= \sum_{\beta \neq h_2}^{N} \frac{\varphi_{mn}(z_1, \ldots, z_M; h_1, \eta_\beta)}{|h_2 - \eta_\beta|^2} + \sum_{\beta \neq h_1}^{N} \frac{\varphi_{mn}(z_1, \ldots, z_M; \eta_\beta, h_2)}{|h_1 - \eta_\beta|^2}$$

$$
\begin{aligned}
&= \sum_{l=0}^{M} \sum_{\beta \neq h_2}^{N} \frac{\eta_\beta^n (\eta_\beta - h_2)^l}{l! |h_2 - \eta_\beta|^2} \frac{\partial^l}{\partial \eta_\beta^l} \left(\frac{\varphi_{mn}(z_1, \dots, z_M; h_1, \eta_\beta)}{\eta_\beta^n} \right) \Bigg|_{\eta_\beta = h_2} \\
&\quad + \sum_{l=0}^{M} \sum_{\beta \neq h_1}^{N} \frac{\eta_\beta^m (\eta_\beta - h_1)^l}{l! |h_1 - \eta_\beta|^2} \frac{\partial^l}{\partial \eta_\beta^l} \left(\frac{\varphi_{mn}(z_1, \dots, z_M; \eta_\beta, h_2)}{\eta_\beta^m} \right) \Bigg|_{\eta_\beta = h_1} \\
&= \sum_{l=0}^{M} \frac{h_1^{m+l}}{l!} B_l^m \frac{\partial^l}{\partial h_1} \left(\frac{\varphi_{mn}}{h_1^m} \right) + \sum_{l=0}^{M} \frac{h_2^{n+l}}{l!} B_l^n \frac{\partial^l}{\partial h_2} \left(\frac{\varphi_{mn}}{h_2^n} \right) \\
&= \left\{ \left(\frac{N^2 - 1}{6} + \frac{m(m-N)}{2} + \frac{n(n-N)}{2} \right) h_1^m h_2^n \right. \\
&\quad - \left(\frac{N-1}{2} - m \right) h_1^{m+1} h_2^n \frac{\partial}{\partial h_1} - \left(\frac{N-1}{2} - n \right) h_1^m h_2^{n+1} \frac{\partial}{\partial h_2} \\
&\quad \left. + \frac{1}{2} h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} + \frac{1}{2} h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2} \right\} \frac{\varphi_{mn}(z_1, \dots, z_M; h_1, h_2)}{h_1^m h_2^n},
\end{aligned} \tag{31}
$$

where we have introduced the coefficients $B_l^n = -\sum_{\beta=1}^{N-1} \eta_\beta^{n+1} (\eta_\beta - 1)^{l-2}$, which are evaluated in Appendix A. (31) is valid if and only if $0 \leq n, m \leq (N+2)/2$, which finally leads to the restriction (22) for the actual Ψ_{mn}^{ho}'s.

For the treatment of the $\downarrow$-charge kinetic term we avail ourselves of the fact that φ_{mn} can be equally expressed by the up-spin or down-spin variables, as we show in Appendix B. If we denote the down-spin coordinates by w_i, we obtain

$$
\begin{aligned}
&\left[H_{\mathrm{C}}^{\downarrow} \varphi_{mn} \right] (z_1, \dots, z_M; h_1, h_2) \equiv \left[\sum_{\alpha \neq \beta}^{N} \frac{P_{\mathrm{G}} c_{\beta\downarrow} c_{\alpha\downarrow}^\dagger P_{\mathrm{G}}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right] (z_1, \dots, z_M; h_1, h_2) \\
&= \left\{ \left(\frac{N^2 - 1}{6} + \frac{m(m-N)}{2} + \frac{n(n-N)}{2} \right) h_1^m h_2^n \right. \\
&\quad - \left(\frac{N-1}{2} - m \right) h_1^{m+1} h_2^n \frac{\partial}{\partial h_1} - \left(\frac{N-1}{2} - n \right) h_1^m h_2^{n+1} \frac{\partial}{\partial h_2} \\
&\quad \left. + \frac{1}{2} h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} + \frac{1}{2} h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2} \right\} \frac{\varphi_{mn}(w_1, \dots, w_M; h_1, h_2)}{h_1^m h_2^n}.
\end{aligned} \tag{32}
$$

Using identities verified in Appendix C for the derivatives with respect to the z_i's and $h_{1,2}$'s, the total charge-kinetic term becomes

$$
\begin{aligned}
&\left[\sum_{\alpha \neq \beta}^{N} \frac{H_{\mathrm{C}}^{\downarrow} + H_{\mathrm{C}}^{\uparrow}}{|\eta_\alpha - \eta_\beta|^2} \varphi_{mn} \right] (z_1, \dots, z_M; h_1, h_2) \\
&= \left\{ \left[\frac{N^2 - 1}{3} + m(m-N) + n(n-N) - C_2 - C_1^2 + m \left(C_1 - \frac{1}{2} \right) \right. \right. \\
&\quad \left. + n \left(C_1 - \frac{1}{2} \right) - \frac{h_1 + h_2}{h_1 - h_2} \left(\frac{m-n}{2} \right) \right] h_1^m h_2^n + h_1^{m+2} h_2^n \frac{\partial^2}{\partial h_1^2} + h_1^m h_2^{n+2} \frac{\partial^2}{\partial h_2^2}
\end{aligned}
$$

$$+ h_1^m h_2^{n+1} \frac{h_2}{(h_1 - h_2)} \frac{\partial}{\partial h_2} + h_1^{m+1} h_1^n \frac{h_1}{h_2 - h_1} \frac{\partial}{\partial h_1} + C_1 h_1^{m+1} h_2^n \frac{\partial}{\partial h_1}$$

$$+ C_1 h_1^m h_2^{n+1} \frac{\partial}{\partial h_2} + h_1^m h_2^n \sum_i^M \frac{h_2^2}{(z_i - h_2)^2} + h_1^m h_2^n \sum_i^M \frac{h_1^2}{(z_i - h_1)^2}$$

$$+ \left. \frac{h_1^2 + h_2^2}{(h_1 - h_2)^2} \right\} \frac{\varphi_{mn}(z_1, \ldots, z_M; h_1, h_2)}{h_1^m h_2^n}, \tag{33}$$

with the constants $C_1 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha) = (N-1)/2$ and $C_2 = \sum_{\alpha=1}^{N-1} 1/(1-\eta_\alpha)^2 = (6N - 5 - N^2)/12$ introduced and evaluated in [4]. Summing up all terms, we finally obtain the action of the Hamiltonian (2) on the auxiliary wave functions φ_{mn}:

$$\begin{aligned} H_{\rm KY} \varphi_{mn} &= \frac{2\pi^2}{N^2} \left\{ \frac{8 - 9N}{8} + m(m-N) + n(n-N) + m\frac{N-2}{2} + n\frac{N-2}{2} \right. \\ &\left. -\frac{1}{2} \frac{h_1 + h_2}{h_1 - h_2}(m-n) + \frac{h_1^2 + h_2^2}{(h_1 - h_2)^2} \right\} \varphi_{mn}. \end{aligned} \tag{34}$$

With (26) this implies

$$\begin{aligned} H_{\rm KY} \Psi_{mn}^{\rm ho} &= \frac{2\pi^2}{N^2} \left[\left(-\frac{8+N}{8} + 1 + \left(m - \frac{N}{2}\right) m + \left(n - \frac{N}{2}\right) n \right) \Psi_{mn}^{\rm ho} \right. \\ &\left. + \frac{m-n}{2} (h_1 - h_2) \frac{h_1 + h_2}{h_1 - h_2} (h_1^m h_2^n - h_1^n h_2^m) \Psi_0 \right] \\ &= \frac{2\pi^2}{N^2} \left[-\frac{N}{8} + \left(m - \frac{N}{2}\right) m + \left(n - \frac{N}{2}\right) n + \frac{m-n}{2} \right] \Psi_{mn}^{\rm ho} \\ &+ \frac{2\pi^2}{N^2} (m-n) \sum_{l=1}^{\lfloor \frac{m-n}{2} \rfloor} \Psi_{m-l,n+l}^{\rm ho}, \end{aligned} \tag{35}$$

where we have used $\frac{x+y}{x-y}(x^m y^n - x^n y^m) = 2\sum_{l=0}^{m-n} x^{m-l} y^{n+l} - (x^m y^n + x^n y^m)$. The symbol $\lfloor\ \rfloor$ denotes the floor function, *i.e.*, $\lfloor x \rfloor$ is the largest integer $l \leq x$. First, note that the action of the Hamiltonian on the $\Psi_{mn}^{\rm ho}$'s is trigonal, *i.e.*, the "scattering" in the last line is only to lower values of $m-n$. Second, (35) shows that the states $\Psi_{mn}^{\rm ho}$ form a non-orthogonal set. We will now proceed to construct an orthogonal basis of energy eigenfunctions.

C. Energy eigenstates

Due to the trigonal structure of the Hamiltonian when acting on the $\Psi_{mn}^{\rm ho}$'s we can derive the energy eigenstates using the Ansatz

$$\left| \Phi_{mn}^{\rm ho} \right\rangle = \sum_{l=0}^{\lfloor \frac{m-n}{2} \rfloor} a_l^{mn} \left| \Psi_{m-l\ n+l}^{\rm ho} \right\rangle, \tag{36}$$

which yields the recursion relation

$$a_l^{mn} = -\frac{1}{2l(l-\frac{1}{2}+n-m)} \sum_{k=0}^{l-1} a_k^{mn}(m-n-2k), \quad a_0^{mn} = 1. \tag{37}$$

This defines the two-holon energy eigenstates (36). The energies are given by

$$E_{mn}^{\text{ho}} = E_0 + \frac{2\pi^2}{N^2}\left[\left(m-\frac{N}{2}\right)m + \left(n-\frac{N}{2}\right)n + \frac{m-n}{2}\right], \tag{38}$$

where the momentum quantum numbers satisfy

$$0 \le n \le m \le \frac{N}{2}, \tag{39}$$

and the total momentum is given by (24).

For the lowest energy state, (38) simplifies (up to an additive constant $\pi^2/12N$) to the ground-state energy of the chain doped with two holes, which is a special case of the result by Kuramoto and Yokoyama [5] for the ground state at general filling fraction.

VII. FRACTIONAL STATISTICS

Fractional statistics in one-dimensional systems was originally introduced by Haldane [8] in the context of non-trivial state counting rules. Recently, it was realized that the fractional statistics of spinons in the HSM also manifests itself in specific quantization rules for the individual spinon momenta [9, 10]. We now apply this line of analysis to the holon excitations in the KYM.

To begin with, let us recall that the asymptotic Bethe ansatz solution of the KYM [6, 12] implies that the holons are free, *i.e.*, that they interact only through their statistics, while there is no position or momentum dependent interaction potential between them. This induces us to rewrite the two-holon energy (38) as

$$E_{mn}^{\text{ho}} = E_0 + \epsilon^{\text{ho}}(p_m) + \epsilon^{\text{ho}}(p_n), \tag{40}$$

where we assume the one-holon dispersion (18) and introduce the single-holon momenta according to

$$p_m = -\frac{\pi}{2} + \frac{2\pi}{N}\left(m+\frac{1}{4}\right), \qquad p_n = -\frac{\pi}{2} + \frac{2\pi}{N}\left(n-\frac{1}{4}\right). \tag{41}$$

Note that the fractional shifts of $\frac{2\pi}{N} \cdot \frac{1}{4}$ in p_m and p_n occur in opposite directions. Since $n \leq m$, the momenta are shifted away from each other, implying $p_m > p_n$. The shifts directly follow from (40); any other assignment of the single particle momenta would yield an additional interaction term in the energy, corresponding to the last term in (38) above. For the difference in the individual holon momenta we hence obtain

$$p_m - p_n = \frac{2\pi}{N} \left(\frac{1}{2} + \text{integer} \right). \tag{42}$$

We interpret this result as a direct manifestation of the half-Fermi statistics of the holons, as the shift in the single particle momenta can be attributed to a statistical phase acquired by the states as the holons pass through each other [10]. Indications of the half-Fermi statistics of the holons have previously been observed in thermodynamical quantities [13, 14] of the KYM as well as the electron addition spectrum [15, 16].

Let us now elaborate on the general implictions of this result. The wave functions we have obtained above are of course eigenstates of the KYM only, which is as idealized as integrable and exactly soluble models tend to be. The quantization rules for the single particle momenta we have obtained for this model, however, have a much broader validity. As mentioned above, the unique feature of the KYM is that the holons are free in the sense that they only interact through their fractional statistics. The single particle momenta of the holons are hence good quantum numbers, which assume fractionally spaced values. For two holons, these are given by (41). The crucial observation in this context is that the statistics of the holons is a quantum invariant and as such independent of the details of the model. This implies directly that the fractional spacings are of universal validity as well. If we were to supplement the model we have studied by a potential interaction between the holons, say a Coulomb potential, this interaction would introduce scattering matrix elements between the exact eigenstates we obtained and labeled according to their fractionally spaced single particle momenta. These momenta would hence no longer constitute good quantum numbers. The new eigenstates would be superpositions of states with different single particle momenta, which individually, however, would still possess the fractionally shifted values. In other words, looking at the quantization condition (42), the "1/2" on the left of the equation will still be a good quantum number, while the "integer" will turn into a "superposition of integers" in the presence of an interaction between the holons.

VIII. CONCLUSIONS

In this article we have studied the two-holon states of the Kuramoto–Yokoyama model. We constructed the explicit two-holon wave functions and derived their momenta and energies. The results display the half-Fermi statistics of the holons, which manifests itself in a shift of $\frac{1}{2}\frac{2\pi}{N}$ in the difference of the individual holon momenta.

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APPENDIX A: B-SERIES

Evaluation of the series

$$B_l^n = -\sum_{\beta=1}^{N-1} \eta_\beta^{n+1}(\eta_\beta - 1)^{l-2}, \tag{A1}$$

with l restricted to $0 \leq l \leq M = (N-2)/2$ yields

$$B_0^n = \frac{N^2-1}{12} + \left(\frac{n(n-N)}{2}\right) \quad \text{for} \quad 0 \leq n \leq N, \tag{A2}$$

$$B_1^n = \begin{cases} n - \dfrac{N-1}{2} & \text{for } 0 \leq n < N, \\ -\dfrac{N-1}{2} & \text{for } n = N, \end{cases} \tag{A3}$$

$$B_2^n = \begin{cases} 1 & \text{for } 0 \leq n \leq N-2,\ n = N, \\ 1-N & \text{for } n = N-1, \end{cases} \tag{A4}$$

$$B_l^n = \begin{cases} 0 & \text{for } l \geq 3,\ 0 \leq n \leq \dfrac{N+2}{2}, \\ N\dbinom{l-2}{N-n-1} & \text{for } l \geq 3,\ \dfrac{N+2}{2} < n \leq N. \end{cases} \tag{A5}$$

Proof: B_0^n, B_1^n, and B_2^n are found by straight forward evaluation of the respective sums. For (A5) consider

$$
\begin{aligned}
B_l^n &= -\sum_{\alpha=1}^{N-1} \eta_\alpha^{n+1} \sum_{k=0}^{l-2} \binom{l-2}{k} (-1)^{l-k-2} \eta_\alpha^k \\
&= \sum_{k=0}^{l-2} \binom{l-2}{k} (-1)^{l-k-1} \Big(1 - \sum_{\alpha=1}^{N} \eta_\alpha^{k+l+1} \Big) \\
&= \begin{cases} -\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} = 0 & \text{for} \quad 3 \leq l, 0 \leq n \leq (N+2)/2, \\ \sum_{k=0}^{l-2} \binom{l-2}{k} N \delta_{k,N-1-n} = N \binom{l-2}{N-n-1} & \text{otherwise.} \end{cases}
\end{aligned}
$$

For the last steps note that the the binomial coefficients of even and odd sites equal each other.

APPENDIX B: WAVE FUNCTION IN ↑- AND ↓-SPIN COORDINATES

The wave functions Ψ_{mn}^{ho} can be equally expressed either in up-spin (z) or down-spin coordinates (w):

$$
\begin{aligned}
&\frac{\Psi_{mn}^{\text{ho}}(z_1, ..., z_M; h_1, h_2)}{h_1^m h_2^n (h_1 - h_2)} \\
&= (-1)^{\frac{1}{2}M(M+1)} \frac{\prod_j^M (h_1 - z_j)(h_2 - z_j) z_j \prod_{i \neq j}^M (z_i - z_j) \prod_{l,j}^M (w_l - z_j)}{\prod_{l,j}^M (w_l - z_j)} \\
&= \frac{\Psi_{mn}^{\text{ho}}(w_1, ..., w_M; h_1, h_2)}{h_1^m h_2^n (h_1 - h_2)}.
\end{aligned} \tag{B1}
$$

This identity applies to the auxiliary wave functions φ_{mn}, as the prepolynomial contains only the coordinates $h_{1,2}$.

APPENDIX C: A DERIVATIVE IDENTITY

The necessary relation for the ↓-charge kinetic term is

$$
\begin{aligned}
&\sum_{i \neq j} \frac{h_2^2}{(w_i - h_2)(w_j - h_2)} \\
&= -C_2 + C_1^2 + 2 \sum_i \frac{h_2^2}{(z_i - h_2)^2} + \sum_{i \neq j} \frac{h_2}{z_i - h_2} \frac{h_2}{z_j - h_2} + 2 \frac{h_2^2}{(h_1 - h_2)^2}
\end{aligned}
$$

$$+2C_1 \sum_i \frac{h_2}{z_i - h_2} + 2C_1 \frac{h_2}{h_1 - h_2} + 2\frac{h_2}{h_1 - h_2} \sum_i \frac{h_2}{z_i - h_2}, \tag{C1}$$

with C_1 and C_2 defined as above. (C1) is also valid for $h_1 \leftrightarrow h_2$.

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[1] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).

[2] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).

[3] M. C. Gutzwiller, Phys. Rev. Lett **10**, 159 (1963).

[4] B. A. Bernevig, D. Giuliano, and R. B. Laughlin, Phys. Rev. B **64**, 024425 (2001).

[5] Y. Kuramoto and H. Yokoyama, Phys. Rev. Lett. **67**, 1338 (1991).

[6] Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. Lett. **73**, 2887 (1994); *ibid.* **74**, E3501 (1995).

[7] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

[8] F. D. M. Haldane, Phys. Rev. Lett. **67**, 937 (1991).

[9] M. Greiter and D. Schuricht, Phys. Rev. B **71**, 224424 (2005).

[10] M. Greiter, submitted to Phys. Rev. Lett.

[11] B. A. Bernevig, D. Giuliano, and R. B. Laughlin, Phys. Rev. B **65**, 195112 (2002).

[12] F. H. L. Eßler, Phys. Rev. B **51**, 13357 (1995).

[13] Y. Kuramoto and Y. Kato, J. Phys. Soc. Jpn. **64**, 1338 (1995).

[14] Y. Kato and Y. Kuramoto, J. Phys. Soc. Jpn. **65**, 1622 (1996).

[15] M. Arikawa, Y. Saiga, and Y. Kuramoto, Phys. Rev. Lett. **86**, 3096 (2001).

[16] M. Arikawa, T. Yamamoto, Y. Saiga, and Y. Kuramoto, Nucl. Phys. B **702**, 380 (2004).